Generalized eigenfunction expansions; an application to linear water waves

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1 Introduction

In this paper, we show a general way of establishing *eigenfunction expansions* of the *transient* state of a linear scattering problem, *i.e.*, its decomposition on a continuous family of *time-harmonic* states which represent the response of the system to a family of time-harmonic plane waves. We illustrate the method in the context of linear water waves.

Since the sixties, the question has been dealt with for many wave propagation phenomena (see the references in [1], and [2] in hydrodynamics). But the common feature of these studies is that most proofs are highly "problem-dependent", that is, a slight change in the definition of the problem requires to adapt most proofs. The purpose of the present paper is to show a more synthetic approach which has the advantage to allow every *compact* perturbation of a "free" wave problem. Our approach is mainly inspired by the book from Weder [3].

We shall denote $L^2_s(\mathbb{R}) := \left\{ v : \mathbb{R} \to \mathbb{C};\ \int_{\mathbb{R}} (1+x^2)^s |v(x)|^2 \, \mathrm{d}x < \infty \right\}$ for $s \in \mathbb{R}$ (with the particular case $L^2_0(\mathbb{R}) = L^2(\mathbb{R})$), which allows to consider $L^2_s(\mathbb{R})$ and $L^2_{-s}(\mathbb{R})$ as dual spaces in the scheme

$$L^2_s(\mathbb{R}) \subset L^2_0(\mathbb{R}) \subset L^2_{-s}(\mathbb{R}) \ \text{ if } s > 0.$$

This means that the integral $\int_{\mathbb{R}} u \, \overline{v}$ can be seen as the scalar product $(\cdot\,,\cdot)$ in $L^2_0(\mathbb{R})$ as well as the duality product $\langle \cdot\,,\cdot \rangle$ between $L^2_{-s}(\mathbb{R})$ and $L^2_s(\mathbb{R})$.

2 Linear water waves

For the sake of simplicity, we shall illustrate the method with the 2D scattering (in a half space) by a fixed rigid immersed body, but the method easily extends to more involved situations such as the sea-keeping problem for an elastic floating body.

The free problem. Let us first describe the "free"problem, *i.e.*, without scatterer (the tilde character will refer to this free situation). We denote by $\tilde{\Omega} := \left\{ X = (x,y) \in \mathbb{R}^2;\ y < 0 \right\}$ the half-space delimited by the free surface $\tilde{F} := \{x = 0\}$. Without outer excitation, the velocity potentiel $\tilde{\varphi} = \tilde{\varphi}(X,t)$ satisfies

$$\Delta \tilde{\varphi} = 0 \ \text{ in } \tilde{\Omega}, \tag{1}$$

$$\partial_t^2 \tilde{\varphi} + \partial_y \tilde{\varphi} = 0 \ \text{ on } \tilde{F}, \tag{2}$$

together with the initial conditions

$$\tilde{\varphi}(0) = g_0 \text{ and } \partial_t \tilde{\varphi}(0) = \dot{g}_0 \text{ on } \tilde{F}. \tag{3}$$

The well-posedness of this problem is easily seen by rewriting (1)–(2) as the following abstract wave equation on $\tilde{u} := \tilde{\varphi}_{|\tilde{F}}$:

$$\partial_t^2 \tilde{u} + \tilde{\boldsymbol{A}} \tilde{u} = 0 \text{ with } \tilde{\boldsymbol{A}} := (\partial_y \tilde{\boldsymbol{H}})_{|\tilde{F}}, \tag{4}$$

where $\tilde{\boldsymbol{H}}$ denotes the "harmonic lifting" from $\tilde{F}$ to $\tilde{\Omega}$, i.e., for $\tilde{v}$ defined on $\tilde{F}$, the function $\tilde{\psi} = \tilde{\boldsymbol{H}}\tilde{v}$ is the solution to $\Delta\tilde{\psi} = 0$ in $\tilde{\Omega}$ and $\tilde{\psi} = \tilde{v}$ on $\tilde{F}$. It may be seen that $\tilde{\boldsymbol{A}}$ actually defines an unbounded positive selfadjoint operator in $L^2_0(\tilde{F})$. Hence the solution to (4)–(3) writes $\tilde{u}(t) = \mathrm{Re}(\exp(-\mathrm{i}\tilde{\boldsymbol{A}}^{1/2} t)\, \tilde{u}_0)$, where $\tilde{u}_0 := g_0 + \mathrm{i}\tilde{\boldsymbol{A}}^{-1/2} \dot{g}_0$. The Fourier transform actually provides a *diagonal* form of this expression in a *generalized spectral basis* of $\tilde{\boldsymbol{A}}$ defined by the functions

$$\tilde{\boldsymbol{w}}_{\lambda,k}(X) = \frac{\mathrm{e}^{\lambda(\mathrm{i}kx+y)}}{\sqrt{2\pi}} \text{ for } \lambda \in \mathbb{R}^+ \text{ and } k = \pm 1, \tag{5}$$

which are time-harmonic solutions to (1)–(2): they represent plane surface waves of frequency $\sqrt{\lambda}$ which propagate towards $k \times \infty$. In the sequel $\tilde{\boldsymbol{w}}_{\lambda,k}$ will denote either the above functions or their restrictions to $\tilde{F}$. Note that $\tilde{\boldsymbol{w}}_{\lambda,k} \in L^2_{-s}(\tilde{F})$ if $s > 1/2$.

Proposition 1. *The projection on the family* $\{\tilde{\boldsymbol{w}}_{\lambda,k}\}$

$$(\tilde{\boldsymbol{U}}\tilde{v})_{\lambda,k} := \langle \tilde{v}, \tilde{\boldsymbol{w}}_{\lambda,k} \rangle_{\tilde{F}} \quad \forall \tilde{v} \in L^2_s(\tilde{F}) \ (s > 1/2), \tag{6}$$

defines (by density) a unitary transformation from $L^2_0(\tilde{F})$ *to the spectral space* $L^2(\mathbb{R}^+ \times \{\pm 1\}) = \{\hat{u}_{\lambda,k}\,;\ \int_{\mathbb{R}^+} \sum_{k=\pm 1} |\hat{u}_{\lambda,k}|^2 \,\mathrm{d}\lambda < \infty\}$. *Moreover* $\tilde{\boldsymbol{U}}$ *diagonalizes* $\tilde{\boldsymbol{A}}$ *in the sense that* $f(\tilde{\boldsymbol{A}}) = \tilde{\boldsymbol{U}}^* f(\lambda)\, \tilde{\boldsymbol{U}}$ *for every bounded function* $f : \mathbb{R}^+ \to \mathbb{C}$, *which can be written more explicitely*

$$f(\tilde{\boldsymbol{A}})\tilde{v} = \int_{\mathbb{R}^+} f(\lambda) \sum_{k=\pm 1} \langle \tilde{v}, \tilde{\boldsymbol{w}}_{\lambda,k} \rangle \, \tilde{w}_{\lambda,k} \,\mathrm{d}\lambda. \tag{7}$$

The latter formula is the *generalized eigenfunction expansion* of $f(\tilde{\boldsymbol{A}})$. For $f(\lambda) = \exp(-\mathrm{i}\lambda^{1/2} t)$, it yields the diagonal form of the solution to (4)–(3).

A simple perturbation. We claim that a similar expansion hold for every *compact perturbation* of the free water wave problem (with a possible additional discrete contribution due to possible trapped modes). Consider the case of an immersed fixed rigid obstacle. We denote by $\Omega \subset \tilde{\Omega}$ the domain exterior to its boundary Γ (so that $\partial\Omega = \tilde{F} \cup \Gamma$). The equations satisfied by the velocity potential are now given by

$$\Delta\varphi = 0 \text{ in } \Omega, \tag{8}$$

$$\partial_t^2 \varphi + \partial_y \varphi = 0 \text{ on } \tilde{F}, \tag{9}$$

$$\partial_n \varphi = 0 \text{ on } \Gamma, \tag{10}$$

as well as initial conditions similar to (3). Exactly as for the free problem, these equations can be expressed as an abstract wave equation of the form (4) which involves the perturbed selfadjoint operator $\boldsymbol{A} := (\partial_y \boldsymbol{H})_{|\tilde{F}}$ instead of $\tilde{\boldsymbol{A}}$, where $\boldsymbol{H}$ is the perturbed harmonic lifting (obtained by inserting the Neumann condition on Γ).

How can one construct a spectral basis for $\boldsymbol{A}$? Simply by considering two kinds of perturbations of the plane waves $\tilde{w}_{\lambda,k}$, written in the form

$$\boldsymbol{w}^{\pm}_{\lambda,k} = \tilde{\boldsymbol{w}}_{\lambda,k} + \boldsymbol{W}^{\pm}_{\lambda,k}. \tag{11}$$

These functions correspond to time-harmonic solutions to (8)–(10) if $W^{\pm}_{\lambda,k}$ satisfies

$$\Delta \boldsymbol{W}^{\pm}_{\lambda,k} = 0 \quad \text{in } \Omega, \tag{12}$$

$$\partial_y \boldsymbol{W}^{\pm}_{\lambda,k} - \lambda\, \boldsymbol{W}^{\pm}_{\lambda,k} = 0 \quad \text{on } \tilde{F}, \tag{13}$$

$$\partial_n \boldsymbol{W}^{\pm}_{\lambda,k} = -\partial_n \tilde{\boldsymbol{w}}_{\lambda,k} \quad \text{on } \Gamma.$$

The sign $+$, respectively $-$, is assigned to *outgoing*, respectively *incoming*, waves, which is specified by means of the standard *radiation condition* at infinity. To be sure that both families (11) actually define generalized spectral bases for $\boldsymbol{A}$, we shall make use of an abstract framework.

3 Abstract Perturbation Result

For the sake of simplicity, we keep the particular functional spaces introduced in the previous sections to present some general results. We denote by $\tilde{\mathcal{A}}$ and $\mathcal{A}$ two *bounded* and positive selfadjoint operators in $L^2_0(\tilde{F})$ (contrary to $\boldsymbol{A}$ and $\tilde{\boldsymbol{A}}$ which are *unbounded*), by $\tilde{\mathcal{R}}(\zeta) := (\tilde{\mathcal{A}} - \zeta)^{-1}$ and $\mathcal{R}(\zeta) := (\mathcal{A} - \zeta)^{-1}$ their respective resolvents (for $\zeta \in \mathbb{C} \setminus \mathbb{R}^+$), and by $\mathcal{D} := \mathcal{A} - \tilde{\mathcal{A}}$.

We assume that we know a spectral basis $\{\tilde{w}_{\lambda,k}\}$ in $L^2_{-s}(\tilde{F})$ of $\tilde{\mathcal{A}}$ in the sense of Proposition 1. The idea is to search a spectral basis $w_{\lambda,k}$ of $\mathcal{A}$ as a perturbation of the latter. It is readily seen that if the following one-sided limits exist,

$$p^{\pm}_{\lambda,k} := - \lim_{\mathbb{C}^{\pm} \ni \zeta \to \lambda \in \mathbb{R}^+} \mathcal{R}(\zeta)\mathcal{D}\tilde{w}_{\lambda,k} \quad \text{where} \quad \mathbb{C}^{\pm} := \{\zeta \in \mathbb{C};\ \pm \mathrm{Im}\, \zeta > 0\},$$

then $w^{\pm}_{\lambda,k} := \tilde{w}_{\lambda,k} + p^{\pm}_{\lambda,k}$ formally satisfy $(\mathcal{A} - \lambda) w^{\pm}_{\lambda,k} = 0$. Under suitable conditions, this formal construction yields two spectral bases of $\mathcal{A}$.

The study of the behavior of $\mathcal{R}(\zeta)$ near $\mathbb{R}^+$ is the object of the so-called *limiting absorption principle*. Noticing that $\mathcal{R}(\zeta) = \tilde{\mathcal{R}}(\zeta)(\mathrm{Id} + \mathcal{D}\tilde{\mathcal{R}}(\zeta))^{-1}$, it is clear that the existence of the limits $\mathcal{R}(\lambda \pm \mathrm{i}0)$ depends on the existence of $\tilde{\mathcal{R}}(\lambda \pm \mathrm{i}0)$, together with the invertibility of $\mathrm{Id} + \mathcal{D}\tilde{\mathcal{R}}(\lambda \pm \mathrm{i}0)$.

Definition 1. The free operator $\tilde{\mathcal{A}}$ is said to satisfy a "strong limiting absorption principle" if $\tilde{\mathcal{R}}(\zeta) := (\tilde{\mathcal{A}} - \zeta)^{-1}$ considered as an operator from $L^2_s(\tilde{F})$ to $L^2_{-s}(\tilde{F})$ has one-sided limits

$$\tilde{\mathcal{R}}(\lambda \pm \mathrm{i}0) := \lim_{\mathbb{C}^{\pm} \ni \zeta \to \lambda} \tilde{\mathcal{R}}(\zeta) \quad \forall \lambda > 0, \tag{14}$$

and these limits satisfy the following property: if $\mathrm{Im}\langle\tilde{\mathcal{R}}(\lambda \pm \mathrm{i}0)\tilde{u}, \tilde{u}\rangle = 0$ for some $\tilde{u} \in L^2_s(\tilde{F})$, then $\tilde{\mathcal{R}}(\lambda \pm \mathrm{i}0)\tilde{u} \in L^2_0(\tilde{F})$ (which means that a non-excited time-harmonic wave must have a finite energy).

Definition 2. $\mathcal{A}$ is called a compact perturbation of $\tilde{\mathcal{A}}$ if $\mathcal{D}$ extends by density to a bounded operator from $L^2_{-s}(\tilde{F})$ to $L^2_s(\tilde{F})$, and $\mathcal{D}\,\tilde{\mathcal{R}}(\lambda \pm \mathrm{i}0)$ are compact operators in $L^2_s(\tilde{F})$ for every $\lambda > 0$.

Then we have (see [1])

Theorem 1. *Assume that the free operator $\tilde{\mathcal{A}}$ satisfies the strong limiting absorption principle of Definition* 1*. Then every compact perturbation $\mathcal{A}$ of $\tilde{\mathcal{A}}$* (*which is assumed to have no point spectrum, otherwise one has to consider the spectrally absolutely continuous part of $\mathcal{A}$*) *satisfies a similar limiting absorption principle, and the one-sided limits of its resolvent are given by*

$$\mathcal{R}(\lambda \pm \mathrm{i}0) = \tilde{\mathcal{R}}(\lambda \pm \mathrm{i}0)(\mathrm{Id} + \mathcal{D}\,\tilde{\mathcal{R}}(\lambda \pm \mathrm{i}0))^{-1}.$$

Moreover both families $w^{\pm}_{\lambda,k} = (\mathrm{Id} - \mathcal{R}(\lambda \pm \mathrm{i}0)\,\mathcal{D})\tilde{w}_{\lambda,k}$, are generalized spectral bases of $\mathcal{A}$ (*in the sense of Proposition* 1).

4 Application to water wave

We show in this section that our water wave problem actually enters the framework of Theorem 1. Since the latter involves bounded operators, we cannot compare directly $\tilde{\boldsymbol{A}}$ and $\boldsymbol{A}$ but an invertible bounded and real function of these operators, namely $\tilde{\mathcal{A}} := \tilde{\boldsymbol{R}}(-\alpha)$ and $\mathcal{A} := \boldsymbol{R}(-\alpha)$ for a fixed $\alpha \in \mathbb{R}^+$. The link between them derives from the following relation (also valid without the tilde):

$$\tilde{\mathcal{R}}(\zeta) = -\zeta^{-1}\left(\mathrm{Id} + \zeta^{-1}\tilde{\boldsymbol{R}}(\zeta^{-1} - \alpha)\right) \text{ for all } \zeta \in \mathbb{C}\backslash]0, \alpha^{-1}[.$$

Limiting absorption for the free problem. The statement of Definition 1 can be verified directly on the resolvent of $\tilde{\boldsymbol{A}}$. The existence of the limits $\tilde{\boldsymbol{R}}(\lambda \pm \mathrm{i}0)$ is a straightforward consequence of the regularity of $\tilde{\boldsymbol{w}}_{\lambda,k}$ with respect to $\lambda > 0$. Indeed the diagonal form of the resolvent which follows from (7) can be written (for every $\tilde{u}, \tilde{v} \in L^2_s(\tilde{F})$)

$$\left(\tilde{\boldsymbol{R}}(\zeta)\tilde{u}\,,\tilde{v}\right) = \int_{\mathbb{R}^+} \frac{\langle\!\langle \tilde{\varPhi}_\lambda, \overline{\tilde{u}} \otimes \tilde{v}\rangle\!\rangle}{\lambda - \zeta}\mathrm{d}\lambda \;\text{ where }\; \tilde{\varPhi}_\lambda = \sum_{k=\pm 1} \overline{\tilde{\boldsymbol{w}}_{\lambda,k}} \otimes \tilde{\boldsymbol{w}}_{\lambda,k}.$$

By (5), we have $\tilde{\varPhi}_\lambda(x, x') = \pi^{-1}\cos\lambda(x - x')$ which is regular. Hence Plemelj formula yields

$$\lim_{\mathbb{C}^\pm \ni \zeta \to \lambda_0 \in \mathbb{R}^+} (\tilde{\boldsymbol{R}}(\zeta)\tilde{u}\,,\tilde{v}) = PV\int_{\mathbb{R}^+} \frac{\langle\!\langle \tilde{\varPhi}_\lambda, \overline{\tilde{u}} \otimes \tilde{v}\rangle\!\rangle}{\lambda - \lambda_0}\mathrm{d}\lambda \pm \mathrm{i}\pi\langle\!\langle \tilde{\varPhi}_{\lambda_0}, \overline{\tilde{u}} \otimes \tilde{v}\rangle\!\rangle. \qquad (15)$$

In other words, the limits $\tilde{\boldsymbol{R}}(\lambda_0 \pm \mathrm{i}0)$ exist and satisfy the integral representation

$$(\tilde{\boldsymbol{R}}(\lambda_0 \pm \mathrm{i}0)\tilde{u})(x) = \int_{\tilde{F}} G_{\lambda_0 \pm \mathrm{i}0}(x, x')\, \tilde{u}(x')\, \mathrm{d}x',$$

where $G_{\lambda_0 \pm \mathrm{i}0} = PV \int_{\mathbb{R}^+} (\lambda - \lambda_0)^{-1} \tilde{\varPhi}_\lambda \mathrm{d}\lambda \pm \mathrm{i}\pi \tilde{\varPhi}_{\lambda_0}$ is the Green function of the free problem. The additional property of Definition 1 follows from the asymptotic behavior of $(\tilde{\boldsymbol{R}}(\lambda \pm \mathrm{i}0)\tilde{u})(x)$ at infinity, which derives from the explicit knowledge of $G_{\lambda \pm \mathrm{i}0}$: for some $\varepsilon > 0$,

$$(\tilde{\boldsymbol{R}}(\lambda \pm \mathrm{i}0)\tilde{u})(x) = \pm 2\mathrm{i}\pi\, (\tilde{\boldsymbol{U}}\tilde{u})_{\lambda, \pm k_x}\, \tilde{\boldsymbol{w}}_{\lambda, \pm k_x}(x) + o(|x|^{-1/2-\varepsilon}), \tag{16}$$

where $k_x = x/|x|$. Using (15), it is now easy to see that if for a given function $\tilde{u} \in L^2_s(\tilde{F})$ with $s > 1/2$,

$$0 = \mathrm{Im}\langle \tilde{\boldsymbol{R}}(\lambda \pm \mathrm{i}0)\tilde{u}, \tilde{u}\rangle = \pm\pi \sum_{k=\pm 1} |(\tilde{\boldsymbol{U}}\tilde{u})_{\lambda,k}|^2\,,$$

then $(\tilde{\boldsymbol{U}}\tilde{u})_{\lambda,\pm 1} = 0$ and (16) thus shows that $\tilde{\boldsymbol{R}}(\lambda \pm \mathrm{i}0)\tilde{u} \in L^2_0(\tilde{F})$.

Compactness of the perturbation. The fact that our particular perturbed problem satisfies Definition 2 derives from the following proposition and the compactness of the canonical injection from $H^{1/2}_{s+\epsilon}(\tilde{F}) := \{v;\ (1 + x^2)^{(s+\epsilon)/2} v \in H^{1/2}(\tilde{F})\}$ to $L^2_s(\tilde{F})$, with $\epsilon > 0$.

Proposition 2. *The operator $\mathcal{D} := \mathcal{A} - \tilde{\mathcal{A}}$ initially defined on $L^2_0(\tilde{F})$ extends to a continuous operator from $L^2_{-s}(\tilde{F})$ to $H^{1/2}_{s+\epsilon}(\tilde{F})$, with $\epsilon > 0$ such that $1/2 < s < 3/2 - \epsilon$.*

Indeed, the definitions of $\mathcal{A}$ and $\tilde{\mathcal{A}}$ show that, for $f \in L^2_0(\tilde{F})$, $u = \mathcal{D}f$ is given by $u = \psi_{|\tilde{F}}$ where $\psi \in W^1(\Omega) := \{\phi;\ (1+|x|^2)^{-1/2}(\log(2+|x|^2))^{-1}\phi \in L^2_0(\Omega)$ and $\nabla\phi \in (L^2_0(\Omega))^2\}$ is the solution to

$$\left|\begin{array}{l} \Delta\psi = 0 \quad \text{in } \Omega \\ \partial_y\psi + \alpha\psi = 0 \quad \text{on } \tilde{F} \\ \partial_n\psi = -\partial_n\tilde{\phi} \quad \text{on } \Gamma \end{array}\right.$$

and $\tilde{\phi} = \tilde{\boldsymbol{H}}\tilde{u}$ with $\tilde{u}(x) = \int_{\tilde{F}} f(y) G_{-\alpha}(x,y)\mathrm{d}y$. Using an integral representation of ψ and the asymptotic behavior of $G_{-\alpha}$, it may be seen that $\psi_{|\tilde{F}} \in H^{1/2}_{s+\epsilon}(\tilde{F})$ and that $\mathcal{D}$ extends to $L^2_{-s}(\tilde{F})$.

References

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