Generalized eigenfunction expansions for linear water waves

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SUMMARY

The present paper is devoted to generalized eigenfunction expansions for scattering problem. A new method for establishing such expansions is proposed and applied to scattering of linear water waves.

1. Introduction

In this paper, we show a general way of establishing *eigenfunction expansions* for linear scattering problems, and illustrate it in the context of linear water waves. From a physical point of view, such an expansion provides the connexion between *transient* and *time-harmonic* motions. More precisely, it leads to decompose the state of the system at every time on a continuous family of time-harmonic states which represent the response of the system to a family of time-harmonic plane waves.

How can one obtain *eigenfunction expansions*? In the absence of scatterer, the Fourier transform is the very tool for this work (§2.). A natural idea for dealing with scatterers consists in considering the problem as a *perturbation* of the former *free* situation. This leads to consider the eigenfunctions of the free problem as *incident* time-harmonic waves, and to search those of the scattering problem by adding a perturbation term representing a *scattered* time-harmonic wave (§3.).

The first application of this approach is due to Ikebe [4] for the Schrödinger equation. In hydrodynamics, the scattering of linear water waves by a fixed body was first studied by Beale [1], and then by different authors (see [3]). The common feature of these studies is that most proofs are highly "problem-dependent", that is, a slight change in the definition of the perturbed problem requires to adapt most proofs. The purpose of the present paper is to show a more synthetic approach which has the advantage to allow general perturbations of a free water wave problem. In fact we shall see how to construct generalized eigenfunction expansions for any "compact perturbation": we shall give the precise meaning of this compactness property. For the sake of simplicity, we shall illustrate the method with the 2D scattering (in a half space) by a fixed rigid immersed body, but the method easily extends to more involved situations such as the seakeeping problem for an elastic floating body. The general ideas are described in [2], and the detailed application to linear water waves is the object of a forthcoming paper.

We shall use the following notation, for $s \in \mathbb{R}$:

$$L^2_s(\mathbb{R}) = \left\{ v : \mathbb{R} \to \mathbb{C};\ \int_{\mathbb{R}} (1+x^2)^s |v(x)|^2 \,\mathrm{d}x < \infty \right\},$$

(with the particular case $L^2_0(\mathbb{R}) = L^2(\mathbb{R})$), which allows to consider $L^2_s(\mathbb{R})$ and $L^2_{-s}(\mathbb{R})$ as dual spaces in the scheme

$$L^2_s(\mathbb{R}) \subset L^2_0(\mathbb{R}) \subset L^2_{-s}(\mathbb{R}) \ \text{ if } s > 0.$$

This means that the integral $\int_{\mathbb{R}} u\bar{v}$ can be seen as the scalar product $(\cdot\,,\cdot)$ in $L^2_0(\mathbb{R})$ as well as the duality product $\langle\cdot\,,\cdot\rangle$ between $L^2_{-s}(\mathbb{R})$ and $L^2_s(\mathbb{R})$.

2. The "free" problem

This paper deals with *localized* perturbations of the linear water wave equations in the half-space $\tilde{\Omega} = \left\{X = (x,y) \in \mathbb{R}^2;\ y < 0\right\}$ delimited by the free surface $\tilde{F} = \{x = 0\}$ (the tilde character will be used for all quantities related to this free situation). Without outer excitation, the velocity potentiel $\tilde{\varphi} = \tilde{\varphi}(X,t)$ satisfies

$$\text{(1)} \qquad \Delta\tilde{\varphi} = 0 \ \text{ in } \tilde{\Omega},$$

$$\text{(2)} \qquad \partial_t^2\tilde{\varphi} + \partial_y\tilde{\varphi} = 0 \ \text{ on } \tilde{F},$$

together with the initial conditions

$$\text{(3)} \qquad \tilde{\varphi}(0) = g_0 \ \text{ and } \ \partial_t\tilde{\varphi}(0) = \dot{g}_0 \ \text{ on } \tilde{F}.$$

These equations are easily solved using a horizontal Fourier transform. We give in Proposition 1 below an abstract interpretation of this well-known result, with is the basis of our perturbation approach.

Let $\tilde{A}$ denote the operator formally defined by $\tilde{A} = (\partial_y\tilde{H})_{|\tilde{F}}$ where $\tilde{H}$ is the "harmonic lifting" from $\tilde{F}$ to $\tilde{\Omega}$, i.e., for $\tilde{v}$ defined on $\tilde{F}$, the function $\tilde{\psi} = \tilde{H}\tilde{v}$ is the solution to

$$\Delta\tilde{\psi} = 0 \ \text{ in } \tilde{\Omega},$$
$$\tilde{\psi} = \tilde{v} \ \text{ on } \tilde{F}.$$

Setting $\tilde{\varphi} = \tilde{H}\tilde{u}$, this allows us to rewrite (1)–(2) as an abstract wave equation

$$\text{(4)} \qquad \partial_t^2\tilde{u} + \tilde{A}\tilde{u} = 0 \ \text{ with } \ \tilde{A} = (\partial_y\tilde{H})_{|\tilde{F}}.$$

It may be easily seen that $\tilde{A}$ actually defines an unbounded positive selfadjoint operator in $L^2_0(\tilde{F})$, and the solution to (4)–(3) writes

(5) $$\tilde{u}(t) = \Re e\left\{ \mathrm{e}^{-\mathrm{i}\tilde{A}^{1/2}t}\,\tilde{u}_0 \right\},$$

where $\tilde{u}_0 := g_0 + \mathrm{i}\tilde{A}^{-1/2}\dot{g}_0$. The Fourier transform actually provides a *diagonal* form of this expression in a *spectral basis* of $\tilde{A}$ defined by the functions

(6) $$\tilde{w}_{\lambda,k}(X) = \frac{\mathrm{e}^{\lambda(\mathrm{i}kx+y)}}{\sqrt{2\pi}} \text{ for } \lambda \in \mathbb{R}^+ \text{ and } k = \pm 1,$$

which are time-harmonic solutions to (1)–(2): they represent plane surface waves of frequency $\sqrt{\lambda}$ which propagate towards $k \times \infty$. In the sequel $\tilde{w}_{\lambda,k}$ will denote either the above functions or their restrictions to $\tilde{F}$. Note that $\tilde{w}_{\lambda,k} \in L^2_{-s}(\tilde{F})$ if $s > 1/2$.

Proposition 1 *The projection on the family* $\left\{\tilde{w}_{\lambda,k}\right\}$

(7) $$(\tilde{U}\tilde{v})_{\lambda,k} := \left\langle \tilde{v}, \tilde{w}_{\lambda,k} \right\rangle_{\tilde{F}} \quad \forall \tilde{v} \in L^2_s(\tilde{F}) \ (s > 1/2),$$

defines (by density) a unitary transformation from $L^2_0(\tilde{F})$ *to the spectral space*

$$L^2(\mathbb{R}^+ \times \{\pm 1\}) = \left\{ \hat{u}_{\lambda,k}\,;\ \int_{\mathbb{R}^+} \sum_{k=\pm 1} \left|\hat{u}_{\lambda,k}\right|^2 \mathrm{d}\lambda < \infty \right\}.$$

Moreover $\tilde{U}$ *diagonalizes* $\tilde{A}$ *in the sense that* $f(\tilde{A}) = \tilde{U}^* f(\lambda)\tilde{U}$ *for every bounded function* $f : \mathbb{R}^+ \to \mathbb{C}$, *which can be written more explicitely*

(8) $$f(\tilde{A})\tilde{v} = \int_{\mathbb{R}^+} f(\lambda) \sum_{k=\pm 1} \langle \tilde{v}, \tilde{w}_{\lambda,k} \rangle\, \tilde{w}_{\lambda,k}\, \mathrm{d}\lambda.$$

The latter formula is the *generalized eigenfunction expansion* of $f(\tilde{A})$. For $f(\lambda) = \exp(-\mathrm{i}\lambda^{1/2}t)$, it yields the diagonal form of (5):

$$\tilde{u}(t) = \Re e \int_{\mathbb{R}^+} \sum_{k=\pm 1} \langle \tilde{u}^{(0)}, \tilde{w}_{\lambda,k} \rangle\, \tilde{w}_{\lambda,k}\, \mathrm{e}^{-\mathrm{i}\sqrt{\lambda}t}\, \mathrm{d}\lambda.$$

We claim that a similar expansion hold for every *compact perturbation* of the free water wave problem (with a possible additional discrete contribution due to possible trapped modes, see e.g. [5]).

3. A simple perturbation

Consider the problem of scattering of water waves by an immersed fixed rigid obstacle. We denote by $\Omega \subset \tilde{\Omega}$ the domain exterior to its boundary Γ (so that $\partial\Omega = \tilde{F} \cup \Gamma$). The equations satisfied by the velocity potential are now given by

(9) $$\Delta\varphi = 0 \text{ in } \Omega,$$

(10) $$\partial_t^2\varphi + \partial_y\varphi = 0 \text{ on } \tilde{F},$$

(11) $$\partial_n\varphi = 0 \text{ on } \Gamma,$$

as well as initial conditions similar to (3). Exactly as for the free problem, these equations can be expressed as an abstract wave equation of the form (4) which involves the perturbed selfadjoint operator $A = (\partial_y H)_{|\tilde{F}}$ instead of $\tilde{A}$, where H is the perturbed harmonic lifting (obtained by inserting the Neumann condition on Γ).
How can one construct a spectral basis for A? Simply by considering two kinds of perturbations of the plane waves $\tilde{w}_{\lambda,k}$, written in the form

(12) $$w^{\pm}_{\lambda,k} = \tilde{w}_{\lambda,k} + W^{\pm}_{\lambda,k}.$$

These functions correspond to time-harmonic solutions to (9)–(11) if $W^{\pm}_{\lambda,k}$ satisfies

(13) $$\Delta W^{\pm}_{\lambda,k} = 0 \text{ in } \Omega,$$

(14) $$\partial_y W^{\pm}_{\lambda,k} - \lambda W^{\pm}_{\lambda,k} = 0 \text{ on } \tilde{F},$$
$$\partial_n W^{\pm}_{\lambda,k} = -\partial_n \tilde{w}_{\lambda,k} \text{ on } \Gamma.$$

The sign $+$, respectively $-$, is assigned to *outgoing*, respectively *incoming*, waves, which is specified by means of the standard *radiation condition* at infinity:

(15) $$\lim_{R\to+\infty} \int_{|x|=R} \left| \partial_{|x|} W^{\pm}_{\lambda,k} \mp \mathrm{i}\lambda W^{\pm}_{\lambda,k} \right|^2 \mathrm{d}y = 0.$$

To be sure that both families (12) actually define generalized spectral bases for A, we shall make use of an abstract framework.

4. Abstract Perturbation Result

For the sake of simplicity, we keep the particular functional spaces introduced in the previous sections to present some general results. We denote by $\tilde{\mathcal{A}}$ and $\mathcal{A}$ two *bounded* selfadjoint operators in $L^2_0(\tilde{F})$ (contrary to A and $\tilde{A}$ which are *unbounded*). We assume that we know a spectral basis $\left\{\tilde{w}_{\lambda,k} \in L^2_{-s}(\tilde{F})\right\}$ of $\tilde{\mathcal{A}}$ in the sense of Proposition 1
Let us first show an intuitive construction of a generalized spectral basis for $\mathcal{A}$ considered as a perturbation of $\tilde{\mathcal{A}}$. We denote their difference $\mathcal{D} := \mathcal{A} - \tilde{\mathcal{A}}$. The idea is to search solutions $w_{\lambda,k}$ to $(\mathcal{A} - \lambda)w = 0$ in the form $w_{\lambda,k} = \tilde{w}_{\lambda,k} + p_{\lambda,k}$. Using the above definition of $\mathcal{D}$ and the fact that $(\tilde{\mathcal{A}} - \lambda)\tilde{w}_{\lambda,k} = 0$, we see that the perturbation term $p_{\lambda,k}$ must satisfy

(16) $$(\mathcal{A} - \lambda)p_{\lambda,k} = -\mathcal{D}\tilde{w}_{\lambda,k}.$$

But this equation is ill-posed if λ belongs to the spectrum of $\mathcal{A}$, which is precisely the situation we are interested in. A way to solve it consists in replacing first $\mathcal{A} - \lambda$ by $\mathcal{A} - \zeta$ for some $\zeta \in \mathbb{C} \setminus \mathbb{R}^+$. Indeed the resolvent

(17) $$\mathcal{R}(\zeta) := (\mathcal{A} - \zeta)^{-1}$$

defines a bounded operator in $L^2_0(\tilde{F})$ since the spectrum of $\mathcal{A}$ is contained in $\mathbb{R}^+$. Then, setting $\mathbb{C}^\pm = \{\zeta \in \mathbb{C};\ \pm\Im m\,\zeta > 0\}$, we can consider both one-sided limits

$$p^\pm_{\lambda,k} := -\lim_{\mathbb{C}^\pm \ni \zeta \to \lambda \in \mathbb{R}^+} \mathcal{R}(\zeta)\mathcal{D}\tilde{w}_{\lambda,k},$$

which formally satisfy (16). The study of the behavior of $\mathcal{R}(\zeta)$ near $\mathbb{R}^+$ is the object of the so-called *limiting absorption principle*. Noticing that

$$\mathcal{R}(\zeta) = \tilde{\mathcal{R}}(\zeta)(\mathrm{Id} + \mathcal{D}\tilde{\mathcal{R}}(\zeta))^{-1},$$

(which is easily deduced from the definition of $\mathcal{D}$), we see that the existence of $\mathcal{R}(\lambda \pm \mathrm{i}0)$ depends on one hand, on a *limiting absorption principle* for the free problem, i.e., the existence of the limits $\tilde{\mathcal{R}}(\lambda \pm \mathrm{i}0)$, on the other hand, on an additional property which ensures the invertibility of $\mathrm{Id} + \mathcal{D}\tilde{\mathcal{R}}(\lambda \pm \mathrm{i}0)$.
More rigorously, let us introduce the following

Definition 2 *The free operator $\tilde{\mathcal{A}}$ is said to satisfy a "strong limiting absorption principle" if $\tilde{\mathcal{R}}(\zeta) := (\tilde{\mathcal{A}} - \zeta)^{-1}$ considered as an operator from $L^2_s(\tilde{F})$ to $L^2_{-s}(\tilde{F})$ has one-sided limits*

$$\tilde{\mathcal{R}}(\lambda \pm \mathrm{i}0) := \lim_{\mathbb{C}^\pm \ni \zeta \to \lambda} \tilde{\mathcal{R}}(\zeta)\ \ \forall \lambda > 0, \tag{18}$$

and these limits satisfy the following property: if $\Im m \left\langle \tilde{\mathcal{R}}(\lambda \pm \mathrm{i}0)\tilde{u}, \tilde{u} \right\rangle = 0$ for some $\tilde{u} \in L^2_s(\tilde{F})$, then $\tilde{\mathcal{R}}(\lambda \pm \mathrm{i}0)\tilde{u} \in L^2_0(\tilde{F})$ (which means that a non-excited time-harmonic wave must have a finite energy).

Definition 3 *$\mathcal{A}$ is called a compact perturbation of $\tilde{\mathcal{A}}$ if $\mathcal{D}$ appears as a bounded operator from $L^2_{-s}(\tilde{F})$ to $L^2_s(\tilde{F})$, and $\mathcal{D}\tilde{\mathcal{R}}(\lambda \pm \mathrm{i}0)$ are compact operators in $L^2_s(\tilde{F})$ for every $\lambda > 0$.*

Then we have (see [2])

Theorem 4 *Assume that the free operator $\tilde{\mathcal{A}}$ satisfies the strong limiting absorption principle of Definition 2. Then every compact perturbation $\mathcal{A}$ of $\tilde{\mathcal{A}}$ (which is assumed to have no point spectrum) satisfies a similar limiting absorption principle, and the one-sided limits of its resolvent are given by*

$$\mathcal{R}(\lambda \pm \mathrm{i}0) = \tilde{\mathcal{R}}(\lambda \pm \mathrm{i}0)(\mathrm{Id} + \mathcal{D}\tilde{\mathcal{R}}(\lambda \pm \mathrm{i}0))^{-1}.$$

Moreover both families

$$w^\pm_{\lambda,k} = (\mathrm{Id} - \mathcal{R}(\lambda \pm \mathrm{i}0)\,\mathcal{D})\tilde{w}_{\lambda,k},$$

are generalized spectral bases of $\mathcal{A}$.

5. The strong limiting absorption principle

5.1 One-sided limits

We now prove the free operator $\tilde{A}$ satisfies the one-sided limits property of definition 2 consequence of the regularity properties of $\{\tilde{w}_{\lambda,k}\}_{\lambda,k}$ with respect to $\lambda > 0$. For a fixed λ, consider the partial projection $\tilde{U}_\lambda : L^2_s(\tilde{F}) \to \mathbb{C}^2$ defined by $\tilde{U}_\lambda \tilde{v} = \langle \tilde{v}, \tilde{w}_{\lambda,.} \rangle$. Then the diagonal expression of the resolvent $\tilde{R}(\zeta)$ which follows from (8) writes

$$\left(\tilde{R}(\zeta)\tilde{u}, \tilde{v}\right) = \int_{\mathbb{R}^+} \frac{(\tilde{U}_\lambda \tilde{u}, \tilde{U}_\lambda \tilde{v})_{\mathbb{C}^2}}{\lambda - \zeta} \mathrm{d}\lambda\,, \tag{19}$$

with $(\tilde{U}_\lambda \tilde{u}, \tilde{U}_\lambda \tilde{v})_{\mathbb{C}^2} = \sum_{k \in \{\pm 1\}} \tilde{U}_\lambda \tilde{u}(k)\ \overline{\tilde{U}_\lambda \tilde{v}(k)}$.
First notice that $\tilde{U}_\lambda \tilde{u}(k)$ is the Fourier transform $\mathcal{F}\tilde{u}$ of $\tilde{u}$. Then to proceed to the limit, we make use of the continuity of the embedding of $L^2_s(\tilde{F})$ in $L^1(\mathbb{R})$ and the hölderian continuity of $\mathcal{F}(L^2_s(I)) = H^s(I)$, the classical Sobolev Space, where $s > 1/2$ and $I \subsetneq \mathbb{R}^+$:

$$\begin{aligned} \lim_{\mathbb{C}^\pm \ni \zeta \to \lambda_0 \in \mathbb{R}^+} (\tilde{R}(\zeta)\tilde{u}, \tilde{v}) \quad &= \quad PV \int_{\mathbb{R}^+} \frac{(\tilde{U}_\lambda \tilde{u}, \tilde{U}_{\mathbb{C}^2} \tilde{v})_\lambda}{\lambda - \lambda_0} \mathrm{d}\lambda \\ &\pm \mathrm{i}\pi (\tilde{U}_{\lambda_0} \tilde{u}, \tilde{U}_{\lambda_0} \tilde{v})_{\mathbb{C}^2} \end{aligned}$$

5.2 Finite energy property

We are going to stick to the second property of definition 2. To prove this property, we shall use an integral representation of the resolvent $\tilde{R}(\zeta)$ of $\tilde{A}$ together with the asymptotic behaviour of its Green's function.
For a given function $\tilde{u} \in L^2_s(\tilde{F})$ and $\zeta \in \mathbb{C} \setminus \mathbb{R}^+$ we establish from (19) and Fubini's theorem, the following integral representation

$$\tilde{R}(\zeta)\tilde{u}(x) = \int_{\tilde{F}} G_\zeta(x, x')\tilde{u}(x')\mathrm{d}x', \tag{20}$$

$$G_\zeta(x, x') := \sum_{k \in \{\pm 1\}} \int_{\mathbb{R}^+} \frac{\overline{\tilde{w}_{\lambda,k}(x)}\ \tilde{w}_{\lambda,k}(x')}{\lambda - \zeta} \mathrm{d}\lambda, \tag{21}$$

hence

$$G_\zeta(x, x') = \frac{1}{\pi} \int_0^{+\infty} \frac{\cos(r\rho)}{\rho - \zeta} \mathrm{d}\rho \tag{22}$$

with $r := |x - x'|$ and where $G_\zeta(\cdot, \cdot)$ is the nothing but the trace on $\tilde{F}$ of the usual Green's function of the sea-keeping problem, noted here G^{2D}_ζ.
As we search for an asymptotic behaviour of the integral representation of $\tilde{R}(\lambda \pm \mathrm{i}0)$, we have to proceed to the limit $\Im m(\zeta) \to 0$ in (20), on either side of the positive real axis. Denoting by $G_{\lambda \pm \mathrm{i}0}$ the one-sided limits of the Green function, we classically obtain

$$G_{\lambda \pm \mathrm{i}0}(x, x') = \pm \mathrm{i} e^{\pm \mathrm{i}\lambda r} + \frac{1}{\pi} \Re e \left(e^{\mathrm{i}\lambda r} E_1(\mathrm{i}\lambda r) \right)\,,$$

where E_1 denotes the exponential integral.
This function has the following asymptotic behaviors

$$(23) \quad G_{\lambda\pm \mathrm{i}0}(x,x') \mp \mathrm{i}e^{\pm \mathrm{i}\lambda r} = -\frac{\ln(r)}{\pi} + o(1)\ , r \to 0\,,$$

$$(24) \quad G_{\lambda\pm \mathrm{i}0}(x,x') \mp \mathrm{i}e^{\pm \mathrm{i}\lambda r} = o(r^{-1})\ , r \to \infty\,.$$

Now we have to establish the asymptotic behaviour of $\tilde{u}^{\pm} := \tilde{R}(\lambda \pm \mathrm{i}0) f$ at infinity to conclude.
One shows simply using (23)–(24) for any f in $L^2_s(\tilde{F})$, the integral representation $\tilde{u}^{\pm}$ has when $|x| \to +\infty$, uniformly in the direction $k_x = \frac{x}{|x|}$, the following asymptotic expansion

$$\tilde{R}(\lambda \pm \mathrm{i}0) f = \pm 2\mathrm{i}\pi \tilde{w}_{\lambda,\pm k_x}(x) \tilde{U} f(\lambda, \pm k_x) + o(|x|^{-s})$$

with $s > 1/2$.
It is now easy to see that if for a given function $f \in L^2_s(\tilde{F})$,

$$0 = \Im m \langle \tilde{R}(\lambda \pm \mathrm{i}0) f, f \rangle = \pm 2\pi \|\tilde{U} f\|^2$$

then necessarily $\tilde{U} f(\lambda, \pm k_x) = 0$ which implies obviously that $R(\lambda \pm \mathrm{i}0) f \in L^2_0(\tilde{F})$.

6. The compact perturbation property

Since definition 3 involves bounded operators, we cannot compare directly $\tilde{A}$ and A but an invertible bounded and real fonction of those operators as $\tilde{\mathcal{A}} = \tilde{R}(-\alpha)$ and $\mathcal{A} = R(-\alpha)$ with $\alpha \in \mathbb{R}^+$. So the spectral analysis of the latter will provide those of $\tilde{A}$ and A. Moreover, to stick to the results exposed in §4., we consider the part of $\mathcal{A}$ spectrally absolutely continuous still denoted $\mathcal{A}$.
First, it can be easily seen that $\tilde{R}(-\alpha)$ satisfy the definition 2 noticing that

$$\tilde{\mathcal{R}}(\zeta) = \left(\tilde{\mathcal{A}} - \zeta\right)^{-1} = -\zeta^{-1}\left(Id + \zeta^{-1}\tilde{R}(\zeta^{-1} - \alpha)\right)$$

for all $\zeta \in \mathbb{C} \backslash]0, 1/\alpha[$.
Then one shows the following technical result

Proposition 5 *The operator $\mathcal{D} := \mathcal{A} - \tilde{\mathcal{A}}$ naturally defined from $L^2_0(\tilde{F})$ to $L^2_0(\tilde{F})$, acts in fact from $L^2_s(\tilde{F})$ to $H^{1/2}_{s+\varepsilon}(\tilde{F})$, with $\varepsilon > 0$ such that $1/2 < s < 3/2 - \varepsilon$.*

This proposition leads to the compact perturbation property noticing that the canonical injection $H^{1/2}_{s+\varepsilon}(\tilde{F}) \hookrightarrow L^2_s(\tilde{F})$ is compact, and noticing that $\mathcal{D}\tilde{\mathcal{R}}(\lambda \pm \mathrm{i}0)$ can be considered as an operator acting from $L^2_{-s}(\tilde{F})$ to $H^{1/2}_{s+\varepsilon}(\tilde{F})$.
We are now going to describe briefly the way to get proposition 5.
First of all we can notice that setting $\tilde{u} = \tilde{\mathcal{A}} f$ and $u = \mathcal{A} f$ is equivalent to

$$\partial_y \tilde{H}(\tilde{u}) + \alpha \tilde{u} = f \quad \text{on } \tilde{F}\,,$$
$$\partial_y H(u) + \alpha u = f \quad \text{on } \tilde{F}\,.$$

Then using the definition of the harmonic liftings $\tilde{H}$ and H the action of $\mathcal{D}$ can be described by the following diagram

$$\mathcal{D} : f \mapsto \tilde{\Phi} \mapsto \partial_n \tilde{\Phi}_{|\Gamma} \mapsto \Psi \mapsto \Psi_{|\tilde{F}}$$

where $\tilde{\Phi} := \tilde{H}\,\tilde{\mathcal{A}}(f)$ and $\Psi := H\,\mathcal{D}(f)$ satisfy

$$\Delta\tilde{\Phi} = 0 \text{ in } \tilde{\Omega}$$
$$\partial_y \tilde{\Phi} + \alpha\tilde{\Phi} = f \text{ on } \tilde{F}$$

$$\Delta\Psi = 0 \quad \text{in } \Omega$$
$$\partial_y \Psi + \alpha\Psi = 0 \quad \text{on } \tilde{F}$$
$$\partial_n \Psi = -\partial_n \tilde{\Phi} \quad \text{on } \Gamma$$

And we know that $\tilde{\Phi}_{|\tilde{F}}$ and Ψ are given by the integral representations

$$(25) \qquad \tilde{\Phi}_{|\tilde{F}}(x) = -\int_{\tilde{F}} f(y) G_{-\alpha}(x,y) \mathrm{d}y$$

$$(26) \quad \Psi(X) = \int_\Gamma \Psi(X')\partial_{n_{X'}} G^{2D}_{-\alpha}(X,X') - \partial_{n_{X'}}\Psi(X') G^{2D}_{-\alpha}(X,X')\mathrm{d}X'$$

for all $X' \in \Omega$.
Henceforth we have to extend the expression (25) to $L^2_s(\tilde{F})$. This is done using formula (23) and showing simply by integrations that $G_{-\alpha}(x,x') = O(1/r^2)$.
Then we must show that $\Psi_{|\tilde{F}} \in H^{1/2}_{s+\varepsilon}(\tilde{F})$. By virtue of (26), we just have to show that $G^{2D}_{-\alpha}$, $\partial_x G^{2D}_{-\alpha}$ and $\partial^2_x G^{2D}_{-\alpha}$ have the same behaviours on $J := \{(X,X') \in (\mathbb{R}^2_-)^2\,; y = 0\,, X' \in \Gamma\}$ as $1/r^2$.

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